Revised Implicit Equal-Weights Particle Filter

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Outline

- particle filter, sample degeneracy
- equal weights, implicit sampling
- implicit equal-weights particle filter

Particle filter

Probability density $p(x)$ represented by weighted ensemble $\{x_i, w_i\}_{i=1}^{N_e}$

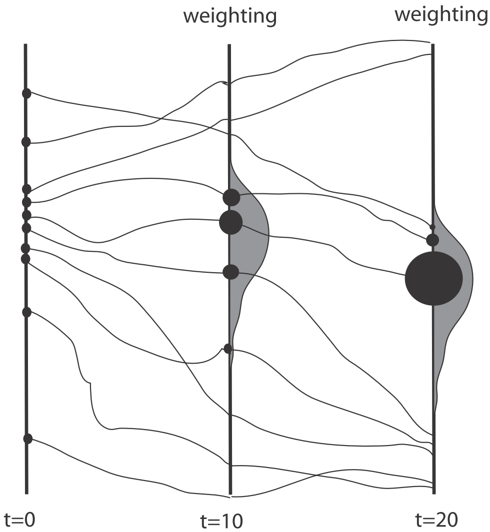
$$\hat{p}(x) = \sum_i w_i \delta(x - x_i)$$

$$E[g(X)] = \int g(x)p(x)dx \approx \int g(x)\hat{p}(x)dx = \sum_i w_i g(x_i)$$

$$p(x|y) = \frac{p(y|x)p(x)}{p(y)} \approx \sum_i w_i^{\text{new}} \delta(x - x_i)$$

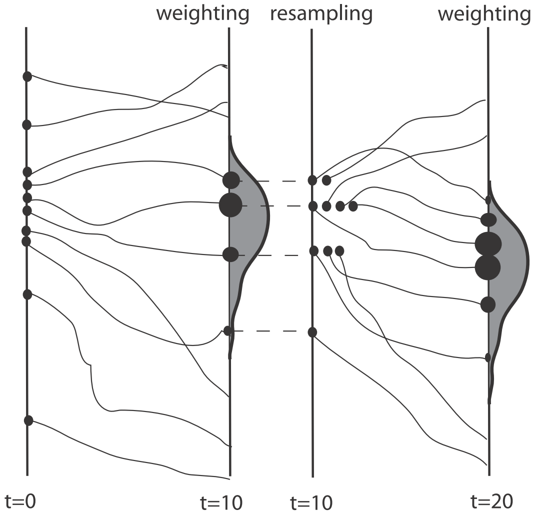
$$w_i^{\text{new}} = w_i \cdot \frac{p(y|x_i)}{p(y)}$$

Importance Sampling (IS) filter



Sequential Importance Resampling (SIR) filter

A.k.a. the **bootstrap filter** (Gordon et al., 1993)



Importance sampling and optimal proposal density

- Draw samples from proposal distribution $q(\mathbf{x})$
- Correct weights for difference between q and p

$$w_i^{\text{corrected}} = \frac{p(\mathbf{x_i})}{q(\mathbf{x}_i)} w_i$$

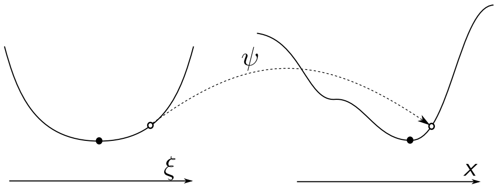
Optimal proposal density

- Suppose target is filtering distribution at time t_n: $p(\mathbf{x}^n|\mathbf{y}^{1:n})$
- Then choosing $q(\mathbf{x}^n) = p(\mathbf{x}^n|\mathbf{x}_i^{n-1}, \mathbf{y}^n)$ minimizes $\mathrm{Var}(w_i^n)$

Implicit sampling

Implicit Particle Filter (IPF) (Chorin and Tu, 2009)

- Want samples from $p(x|y)$
- $\xi \sim g(\xi)$
- ψ maps mode of $g(\xi)$ to mode of $p(x|y)$



- $G(\xi) = -\log g(\xi)$, $F(x) = -\log[p(x|x_{\text{prev}})p(y|x)]$
- To find x given ξ, solve $F(x) - \varphi_F = G(\xi) - \varphi_G$
- $w(x^n) = \frac{p(x^n|x^{n-1})p(y^n|x^{n-1})}{g(\xi)} \left|\frac{\partial x^n}{\partial \xi}\right| \propto e^{-\varphi_F+\varphi_G} \left|\frac{\partial x^n}{\partial \xi}\right|$

Equal Weights

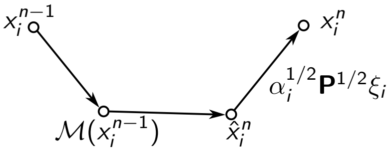
Force all the weights to be equal by construction

$$w_1 = w_2 = \ldots = w_{N_e} = w_{\text{target}}$$

- Transformation $\psi : \xi \mapsto x_i$ involves parameter α_i
- Weight w_i is a function of α_i
- Choose α_i so that $w_i = w_{\text{target}}$

Implicit equal-weights particle filter (IEWPF)

Zhu, van Leeuwen and Amezcua (2016)

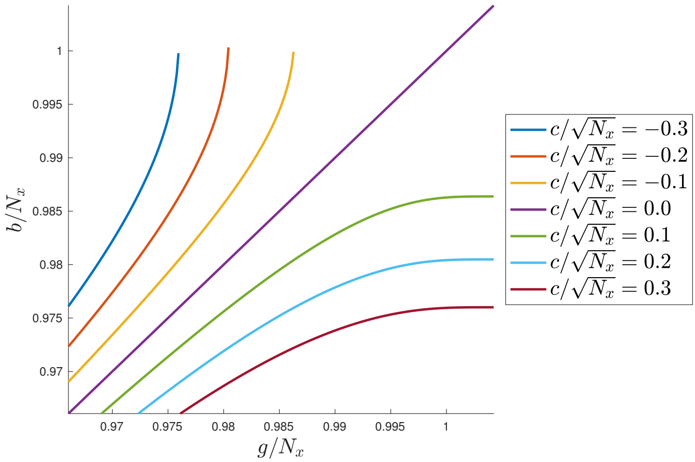


- $x_i^a = \arg\max_x p(x|x_i^{n-1}, y^n)$
- $\xi \sim q(\xi)$
- Weight of particle i:

$$w_i = w_i^{\text{prev}} \cdot \frac{p(x_i^n|x_i^{n-1}, y^n)p(y^n|x_i^{n-1})}{q(\xi)} \left|\frac{\partial x}{\partial \xi}\right| = w_{\text{target}}$$

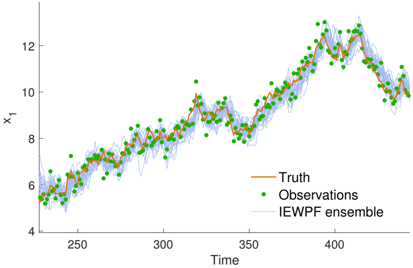
- Solve for α_i to determine x_i^n for $i = 1, \ldots, N_e$

Transformation from ξ to x



$$g = \xi^T \xi, \qquad b = \alpha g$$

Gauss-linear test case



$$\mathbf{x}^n = \mathbf{x}^{n-1} + \boldsymbol{\eta}^{n-1}$$

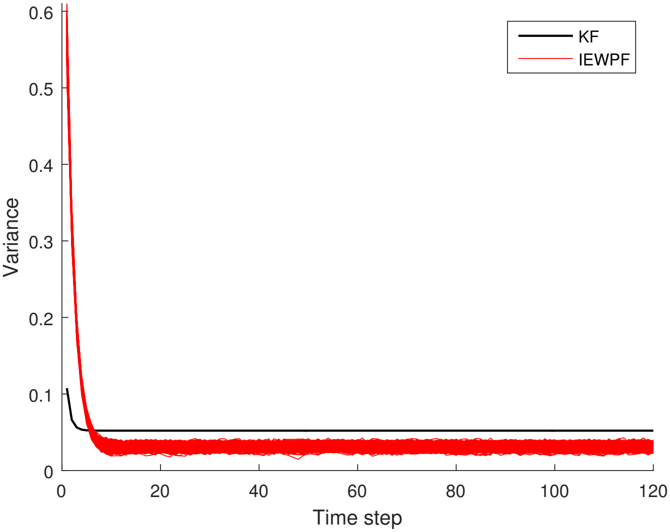
$$\mathbf{y}^n = \mathbf{x}^n_{\text{truth}} + \boldsymbol{\epsilon}^n$$

$$N_x = 1000$$

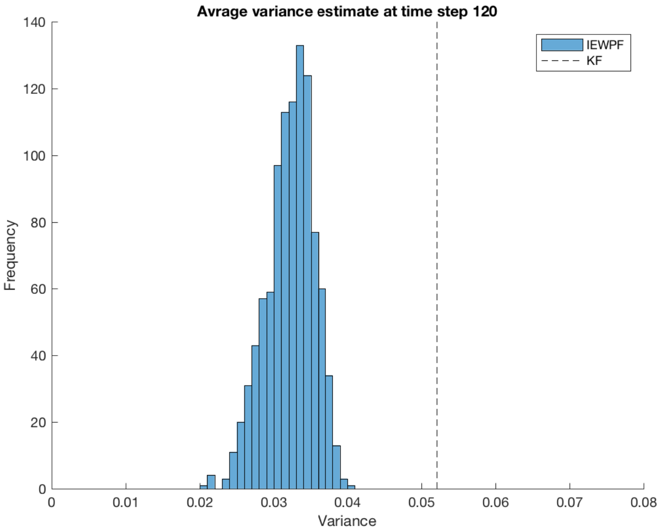
$$N_e = 25$$

$$\boldsymbol{\eta}^{n-1} \sim N(0, \mathbf{Q}), \quad \boldsymbol{\epsilon}^n \sim N(0, \mathbf{R}), \quad \mathbf{x}^0 \sim N(0, \mathbf{B})$$

Gauss-linear test case: Ensemble variance over time



Gauss-linear test case: Ensemble variance final distribution

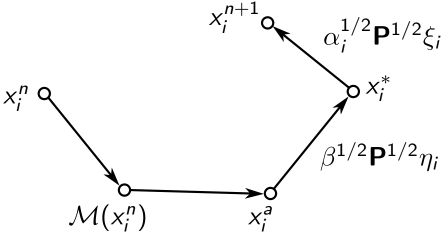


Two-stage IEWPF

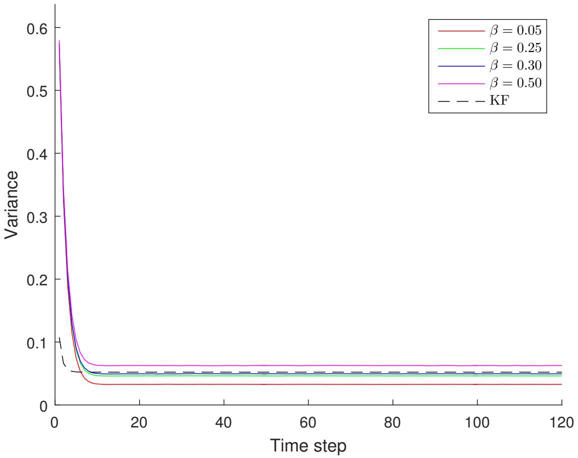
Two-stage update scheme:

$$\mathbf{x}_i^n = \mathbf{x}_i^a + \beta^{1/2}\mathbf{P}^{1/2}\boldsymbol{\eta}_i + \alpha_i^{1/2}\mathbf{P}^{1/2}\boldsymbol{\xi}_i$$

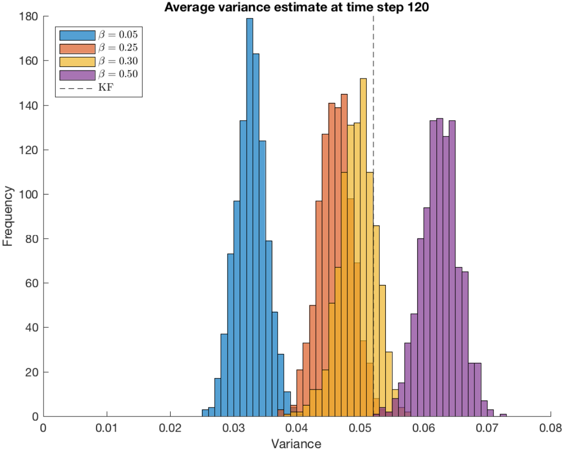
$$\boldsymbol{\xi}_i^T\boldsymbol{\eta}_i = 0$$



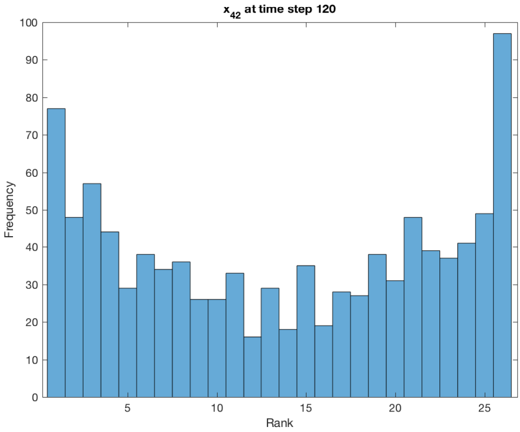
Two-stage IEWPF: Ensemble variance over time



Two-stage Ensemble Variance Final Distribution



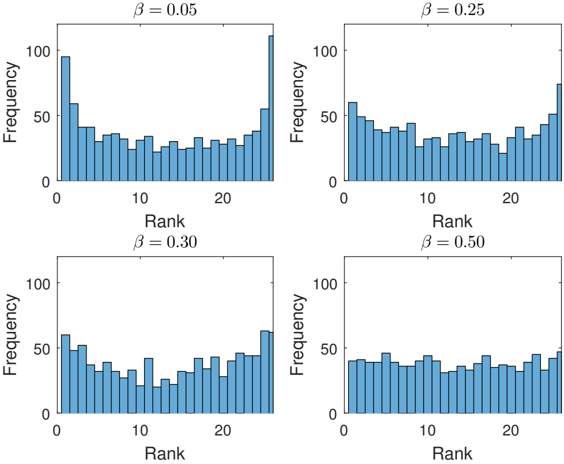
Single-stage IEWPF rank distribution



$$x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(r)} \leq x_{\text{true}} \leq x_{(r+1)} \leq \ldots \leq x_{(N_e)}$$

(should be more or less uniform)

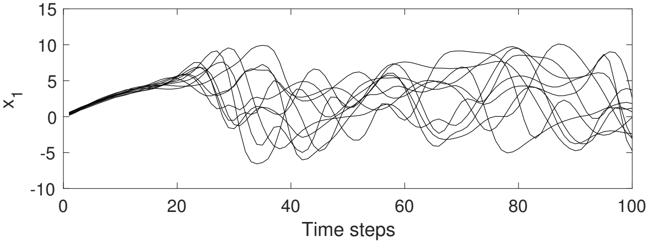
Two-stage IEWPF rank distribution



Non-linear test case

Lorenz96 model with $N_x = 40$, $N_y = 20$, $N_e = 100$

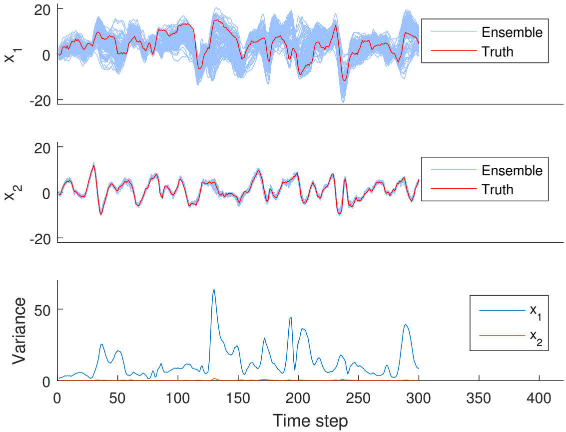
$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -x_{i-2}x_{i-1} + x_{i-1}x_{i+1} - x_i + F, \qquad i = 1, \ldots, N_x.$$



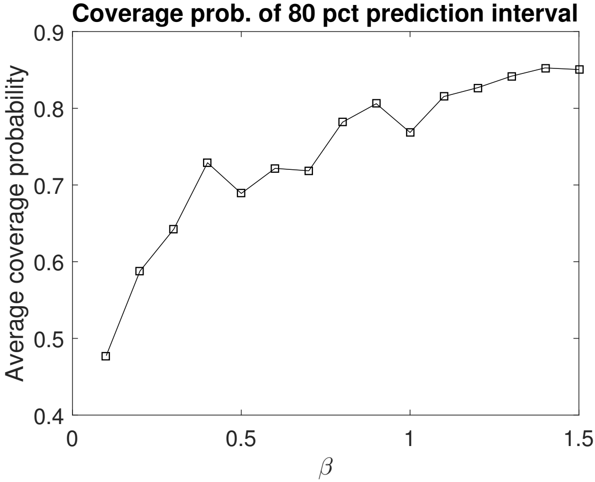
$$\mathbf{x}^n = \mathcal{M}(x^{n-1}) + \boldsymbol{\eta}^{n-1}, \qquad \boldsymbol{\eta}^{n-1} \sim N(0, \mathbf{Q})$$

$$\mathbf{y}^m = \mathbf{H}\mathbf{x}^m + \boldsymbol{\epsilon}^n, \qquad \boldsymbol{\epsilon}^n \sim N(0, \mathbf{R})$$

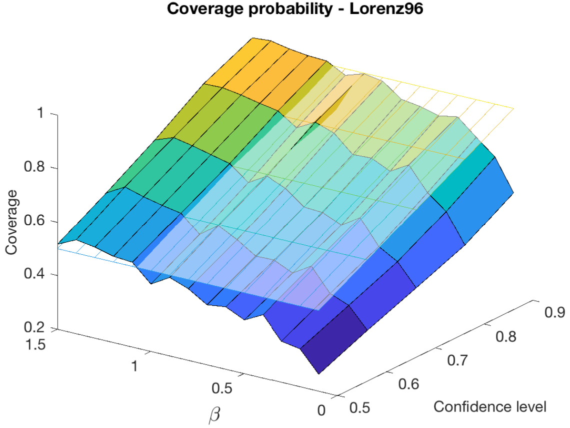
Non-linear test case



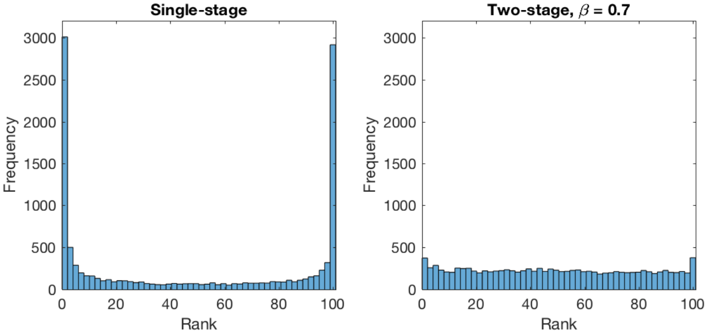
Non-linear test case: Coverage probability



Non-linear test case: Calibration



Non-linear test case: Rank distribution



Conclusion

- IEWPF ensures equal weights, prevents ensemble degeneracy, but underestimates variance
- Two-stage scheme is able to achieve correct variance, but adds a tuning parameter

Properties under study

- Choice of target weight affects quality of estimates
- Setting target weight too large means some particles must get lower weights
- Setting target weight low enough for all weights to be equal induces a bias